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Topological methods for algebraic specification

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Abstract

We introduce an algebraic construction for the Hausdorff extension $H(A)$ of a many-sorted universal algebra A with respect to a family T of Hausdorff topologies on the carrier sets of A. This construction can be combined with other algebraic constructions, such as the initial model construction, to provide methods for the algebraic specification of uncountable algebras, e.g. algebras of reals, function spaces and streams.

1. Introduction

Topology is often regarded as being complementary to algebra in its scope. The idea that these two subjects might therefore be usefully combined is not new. Nineteenth century mathematics saw the introduction of groups of continuous transformations in the work of S. Lie. The theories of topological groups, rings, fields and vector spaces have since been extensively developed, a recent survey is [19].

Universal algebra provides the mathematical foundation for the theory of abstract data types and algebraic specification methods. Therefore it is not surprising to find that *topological universal algebra* can also contribute in this area. The theory of topological universal algebra was introduced by von Dantzig [17] and Birkhoff [2] and has been further developed by for example Markov [9], Pontrjagin [13], Malcev [8], Taylor [16] and Arkhangel'skii [1]. In this paper we present a general study of the topological construction of *closure*, applied to universal algebras.

Topological closure allows us to extend the carrier set of an algebra A by adding all limit points of A in a topology T on a set B which contains. A. We may think of these limit points as "ideal elements" which can be used to solve certain types of equations over the signature of A. For example, let $\mathbf{Q}$ be the ring of rationals and $\mathbf{R}$ be the set of real numbers and let E be the usual Euclidean metric topology on $\mathbf{R}$. The closure of the carrier set of $\mathbf{Q}$ in E is $\mathbf{R}$ itself. (In this case we say that the subset $\mathbf{Q}$ is *dense* in E.)

Now one can consider extending each operation f_A of the algebra A to the limit points in B. If we can find an extension $\hat{f}$ for each f_A which is continuous with respect

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to the topology T (or the appropriate subspace topology if A is not dense in B), then these extensions, together with the original constants of A and the closure $Cl(A)$ of the carrier of A, form a *closed continuous extension* $\hat{A}$ of A. We will show, using basic topology, that if T is Hausdorff then $\hat{A}$ is the *unique* closed continuous extension of A. Thus, when T is Hausdorff, we term the unique closed continuous extension $H(A)$ of A, if it exists, the *Hausdorff extension* of A.

In the case of the ring $\mathbf{Q}$ of rationals, recalling that every metric space is Hausdorff, the Hausdorff extension $H(\mathbf{Q})$ with respect to E exists, namely the ring $\mathbf{R}$ of real numbers. Observe that in this case $H(\mathbf{Q})$ satisfies the ring axioms, which are equational. Inside the ring $\mathbf{R}$ we may also solve equations such as

$$x \cdot x = 2.$$

A fundamental property of *every* Hausdorff extension, that we will establish by means of an algebraic construction of $H(A)$, is that the equations valid in A and $H(A)$ are the same. In particular, if A arises in the context of data type theory as some data type which is a model of an equational specification (Σ, E) then $H(A)$ is a model of the same specification.

We view the construction of Hausdorff extensions as a useful additional method in the area of algebraic specification. Computations in certain specialist fields such as numerical analysis, hardware design, neural networks and dynamical systems, control systems theory and mathematical logic make use of uncountable algebras as *idealised specifications*. Such specifications are *approximated* to some degree of accuracy by actual implementations. Examples include algebras based on:

(i) real or complex valued functions on $\mathbf{R}$ and $\mathbf{C}$,

(ii) transformations on discrete, dense and continuous stream spaces

$$[\mathbf{N} \to \mathbf{A}], \qquad [\mathbf{Q}^+ \to \mathbf{A}], \qquad [\mathbf{R}^+ \to \mathbf{A}],$$

(iii) operators on function spaces $[\mathbf{R}^m \to \mathbf{R}^n]$, $[\mathbf{C}^m \to \mathbf{C}^n]$,

(iv) higher-order functions.

Clearly, by cardinality constraints, such algebras cannot be given a countable algebraic specification using any of the familiar term model constructions such as first-order initial semantics [6], first-order final semantics [18] or higher-order initial semantics [10, 12]. However, given an appropriate separable topological space, (i.e. a space with a countable dense subspace) we may be able to give a recursive (or even finite!) equational specification of a countable dense subalgebra and then construct the (uncountable) Hausdorff extension of a countable term model.

In this paper we introduce a construction for the Hausdorff extension (when it exists) of a (many-sorted) universal algebra with respect to an arbitrary (many-sorted) Hausdorff space. We consider sufficient conditions which guarantee existence for various topologies. We show how the construction can be refined in the case that the given topology satisfies stronger separation or countability axioms. We also illustrate the Hausdorff extension construction by applying it to a case study of second-order algebras including algebras of infinite streams.

The structure of the paper is as follows. In Section 2 we briefly review elementary concepts and results from universal algebra and topology. In Section 3 we introduce the Hausdorff extension construction in the context of closed continuous extensions over arbitrary topological spaces. We consider the existence, uniqueness and construction of Hausdorff extensions. We also consider refinements of the construction in the presence of strong separation and countability conditions. Finally in Section 4 we consider Hausdorff extensions of second-order algebras as an illustrative example of the techniques introduced in Section 3.

We have attempted to make the paper mostly self-contained although the reader will benefit from some prior knowledge of universal algebra and topology. For further information and background results in these areas we suggest [7] for topology and [3] for universal algebra. Surveys of these two subjects oriented towards computer science are [14] for topology and [11] for universal algebra. A recent survey of algebraic specification techniques is [20].

2. Universal algebra and topology

In this section we review some basic definitions, constructions and results of universal algebra, topology and topological universal algebra.

We begin by fixing our notation for many-sorted universal algebra.

Definition 2.1. A *many-sorted signature* is a pair (S, Σ) consisting of:

(i) A non-empty set S. An element $s \in S$ is termed a *sort* and S is termed a *sort set.*

(ii) An $S^* \times S$-indexed family

$$\Sigma = \langle \Sigma_{w,s} \mid w \in S^*, s \in S \rangle$$

of sets of constant and operation symbols. For the empty word $\lambda \in S^*$ and any sort $s \in S$, an element

$$c \in \Sigma_{\lambda,s}$$

is termed a *constant symbol* of sort s. For each non-empty word $w = s(1) \ldots s(n) \in S^+$ and any sort $s \in S$, an element

$$f \in \Sigma_{w,s}$$

is termed an *operation* or *function symbol* of *domain type w*, *codomain type s* and *arity n.* When no ambiguity arises we may use Σ as the name of the signature (S, Σ) and refer to Σ as an *S-sorted signature.*

Definition 2.2. Let Σ be an S-sorted signature. An (S-sorted) Σ *algebra* is a pair (A, Σ^A) consisting of:

(i) An S-indexed family

$$A = \langle A_s \mid s \in S \rangle$$

of non-empty sets. For each sort $s \in S$, the set A_s is termed the *carrier* of sort s.

(ii) An $S^* \times S$-indexed family

$$\Sigma^A = \langle \Sigma^A_{w,s} \mid w \in S^*, s \in S \rangle$$

of sets of constants and operations. For $\lambda \in S^*$ and any sort $s \in S$,

$$\Sigma_{\lambda,s} = \langle c_A \mid c \in \Sigma_{\lambda,s} \rangle,$$

where $c_A \in A_s$ is termed a constant of sort $s \in S$ which interprets the constant symbol $c \in \Sigma_{\lambda,s}$. For each non-empty word $w = s(1) \ldots s(n) \in S^+$ and $s \in S$,

$$\Sigma^A_{w,s} = \langle f_A \mid f \in \Sigma_{w,s} \rangle,$$

where $f_A : A^w \to A_s$ is a total function with domain $A^w = A_{s(1)} \times \cdots \times A_{s(n)}$, codomain A_s and arity n, which interprets the function symbol $f \in \Sigma_{w,s}$. When no ambiguity arises we may use A as the name of the Σ algebra (A, Σ^A).

If $A = \langle A_s \mid s \in S \rangle$ and $B = \langle B_s \mid s \in S \rangle$ are S-indexed families of sets then the basic set-theoretic operations extend pointwise to A and B. Thus we let $A \subseteq B$ denote pointwise inclusion, $A_s \subseteq B_s$ for each $s \in S$. Similarly $A \cap B$, $A \cup B$ and $A = B$ will denote pointwise intersection, union and equality. We use $f : A \to B$ to denote an S-indexed family of mappings $\langle f_s : A_s \to B_s \mid s \in S \rangle$. We let $f(A)$ denote the family $\langle f(A)_s = f_s(A_s) \mid s \in S \rangle$.

Henceforth we will assume the reader is familiar with the basic definitions and results of universal algebra including the notations of Σ subalgebra, Σ congruence and quotient Σ algebra, Σ homomorphism, isomorphism and embedding, Σ equation and equational class or variety of Σ algebras. A special case of the direct product, the *direct power*, will be used extensively in Section 3. We recall this construction here.

Definition 2.3. Let Σ be an S-sorted signature, let A be a Σ algebra and let I be any non-empty set. The *direct power* A^I of A is the Σ algebra with carrier sets

$$A^I_s = [I \to A_s]$$

for each $s \in S$ (i.e. A^I_s is the set of all total functions from I to A_s, or I-indexed vectors from A_s). For each sort $s \in S$ and each constant symbol $c \in \Sigma_{\lambda,s}$ we define

$$c_{A^I}(i) = c_A$$

for each $i \in I$. For each $w = s(1) \ldots s(n) \in S^+$, each $s \in S$, each operation symbol $f \in \Sigma_{w,s}$ and any $a_j \in A^I_{s(j)}$ for $1 \leqslant j \leqslant n$ we define

$$f_{A^I}(a_1, \ldots, a_n)(i) = f_A(a_1(i), \ldots, a_n(i))$$

for each $i \in I$.

We define the S-indexed family $\delta : A \to A^I$ of *diagonal mappings* by

$$\delta_s(a)(i) = a$$

for each sort $s \in S$ and each $a \in A_s$.

It is easily shown that the family $\delta : A \to A^I$ of diagonal mappings is a Σ embedding of A in A^I. Next we introduce a notation for many-sorted topological spaces.

Definition 2.4. Let S be a non-empty set. By an *S-indexed family (A, T) of topological spaces* we mean a family

$$(A, T) = (\langle A_s \mid s \in S\rangle, \langle T_s \mid s \in S\rangle)$$

where for each $s \in S$, A_s is a set and T_s is a topology on A_s, i.e. a collection of subsets of A_s satisfying:

(i) if $F \subseteq T_s$ then $\bigcup F \in T_s$,
(ii) if $U, U' \in T_s$ then $U \cap U' \in T_s$,
(iii) $\emptyset \in T_s$ and $A_s \in T_s$.

The members of T_s are termed *open sets* on A_s. For any $s \in S$ and $a \in A_s$ an open set $U \in T_s$ such that $a \in U$ is termed a *neighbourhood* of a. We let $Nbd_A(a)$ denote the set of all neighbourhoods of a in A. Given an S-indexed family of subsets $B \subseteq A$, for any $s \in S$ an element $a \in A_s$ is said to be *adherent* to B if, and only if, for every neighbourhood $U \in Nbd_A(a)$, we have $U \cap B_s \neq \emptyset$. The set of all points $a \in A_s$ adherent to B_s is termed the *closure* of B_s. We let $Cl(B_s)$ denote the closure of B_s and $Cl(B)$ denotes the family $\langle Cl(B)_s = Cl(B_s) \mid s \in S\rangle$. The subset B_s is said to *closed* if, and only if, $B_s = Cl(B_s)$, and B_s is said to be *dense* in T_s if, and only if, $Cl(B_s) = A_s$.

For any $w = s(1) \ldots s(n) \in S^+$ we let T^w denote the product topology on the cartesian product $A^w = A_{s(1)} \times \cdots \times A_{s(n)}$ with subbasic open sets

$$\langle U \rangle = \{(a_1, \ldots, a_n) \in A^w \mid a_i \in U\}$$

for each open $U \in T_{s(i)}$ and each $1 \leqslant i \leqslant n$. We let $\langle U_1, \ldots, U_n\rangle$ denote the finite intersection $\langle U_1\rangle \cap \cdots \cap \langle U_n\rangle$. For any n-tuple $(a_1, \ldots, a_n) \in A^w$ we let $Nbd_A(a_1, \ldots, a_n)$ denote the set of all neighbourhoods of $(a_1, \ldots, a_n)$ in T^w.

For any $s \in S$ and function $f : A^w \to A_s$, f is said to be *continuous* with respect to T if, and only if, for each open set $U \in T_s$,

$$f^{-1}(U) \in T^w,$$

i.e. inverse images of open sets are open.

Let A be an S-sorted Σ algebra and T be an S-indexed family of topological spaces over the carrier sets of A. Then the pair (A, T) is a *topological Σ algebra* if, and only if, for each $w \in S^+$, $s \in S$ and $f \in \Sigma_{w,s}$, the operation $f_A : A^w \to A_s$ is continuous with respect to T. We let $TopAlg(\Sigma)$ denote the class of all topological Σ algebras.

Most definitions and properties from topology can be extended pointwise from a topological space to an S-indexed family of topological spaces. For example in Section 3 we require the notions of *subspace* and *Hausdorff space*. We make these concepts precise in the many-sorted case.

Definition 2.5. Let (A, T) and (B, T') be S-indexed families of topological spaces.

(i) We say that (A, T) is a *subspace* of (B, T') if, and only if, $A \subseteq B$ and

$$T_s = \{U \cap A_s \mid U \in T'_s\}.$$

(ii) We say that (A, T) is a *Hausdorff space* if, and only if, for each sort $s \in S$ the topology T_s is Hausdorff, i.e. for any pair of elements $a, a' \in A_s$, if $a \neq a'$ then there exist neighbourhoods $U \in Nbd_A(a)$ and $V \in Nbd_A(a')$ such that

$$U \cap V = \emptyset.$$

Clearly if (A, T) is Hausdorff and (B, T') is a subspace of (A, T) then (B, T') is Hausdorff.

The notion of *convergence* is central to topology. A concise way to describe convergence properties is by using *filterbases*. (Another more algebraic approach is to use *nets*. We introduce this latter approach in Section 3.)

Definition 2.6. Let A be a non-empty set. A *filterbase* F on A is a non-empty family of subsets of A such that:
(i) for each $X \in F$, $X \neq \emptyset$,
(ii) for any $X_1, X_2 \in F$ there exists $X_3 \in F$ such that $X_3 \subseteq X_1 \cap X_2$.
A filterbase F on A is a basis for a filter $\hat{F}$ on A by taking

$$\hat{F} = \{Y \subseteq A \mid Y \supseteq X \text{ for some } X \in F\}.$$

By abuse of notation, if $A' \subseteq A$ is any dense subset we let $A' \cap F$ denote the collection of sets

$$A' \cap F = \{A' \cap X \mid X \in F\}.$$

Then $A' \cap F$ is also a filterbase. If B is any non-empty set and $f: A \to B$ is any map then the image of F under f,

$$f(F) = \{f(X) \mid X \in F\}$$

is a filterbase on B. If T is a topology on A and $a \in A$ then the set $Nbd_A(a)$ of all neighbourhoods of a is a filterbase on A termed the *neighbourhood filterbase* of a.

Given a topology T on A, a filterbase F on A *converges* to an element $a \in A$ (with respect to T) if, and only if, for each neighbourhood $U \in Nbd_A(a)$ there exists $X \in F$ such that $X \subseteq U$.

3. Hausdorff extensions

In this section we consider the existence, uniqueness and construction of closed continuous extensions of a many-sorted algebra A with respect to a family T of topological spaces. For reasons already indicated in the introduction, we will concentrate on the important special case where T is a family of Hausdorff spaces.

3.1. Closed continuous extensions in topological spaces

Let Σ be an S-sorted signature and let A be a Σ algebra. In the context of algebraic specification A will typically arise as some model of an axiomatic specification (Σ, Φ), where Φ is some set of logical formulas (for example equations) over Σ. Let (B, T) be an S-indexed family of topological spaces such that $A \subseteq B$. For each sort $s \in S$, we may form the topological closure $Cl(A_s)$ of the carrier set A_s in T_s. The subset $Cl(A_s) \subseteq B_s$ may be given the subspace topology induced on T_s. Now $A_s \subseteq Cl(A_s)$ and if A_s is not closed in T_s then $A_s \subset Cl(A_s)$. This topological construction on the carrier sets of the algebra A adds the adherent elements of B_s to A_s as new data elements. These elements can be regarded as limit points of sequences (or more generally nets) of elements of A_s. The problem arises to construct a Σ algebra C which is a *closed continuous extension* of A in the sense that C extends A, for each sort $s \in S$ the carrier set C_s is the closure $Cl(A_s)$ of A_s in T_s, and each operation f_C of C is continuous. We make these concepts precise as follows.

Definition 3.1. Let Σ be an S-sorted signature, let A and C be Σ algebras and let (B, T) be an S-indexed family of topological spaces such that $A \subseteq B$. We say that C is a *closed continuous extension* of A with respect to (B, T), if, and only if:

(i) C extends A,

(ii) for each sort $s \in S$, $C_s = Cl(A_s)$, and

(iii) for each $w \in S^+$, $s \in S$ and operation symbol $f \in \Sigma_{w,s}$, the operation

$$f_C : C^w \to C_s$$

is continuous in the induced subspace topology (C, T') on T.

Since it can be easily answered using elementary results from topology, we next address the question of uniqueness of closed continuous extensions.

3.2. Uniqueness of Hausdorff extensions

Given an S-sorted signature Σ, a Σ algebra A and an S-indexed family (B, T) of topological spaces such that $A \subseteq B$ we consider the following

Uniqueness Problem. *Find sufficient conditions on the family T of topologies such that a closed continuous extension of A, if it exists, is unique.*

In general there may exist many different closed continuous extensions of A with respect to (B, T). However, a solution to the uniqueness problem can be easily obtained from elementary topology. We consider the situation in which T is a family of Hausdorff topologies. Then a simple consequence of a Hausdorff separation axiom ensures that a closed continuous extension of A with respect to (B, T) (if it exists) is unique. The basic fact we use is the following.

Proposition 3.2. *If T is a family of Hausdorff topologies and A is a dense in T then for any $w \in S^+$, any $s \in S$ and any continuous functions $f, g : B^w \to B_s$,*

$$f = g \Leftrightarrow f|_A = g|_A,$$

where $f|_A$ and $g|_A$ are the restrictions to A of f and g respectively.

Proof. $(\Rightarrow)$ Trivial since $f|_A$ and $g|_A$ are restrictions.

$(\Leftarrow)$ We prove the contrapositive. Suppose $f \neq g$. Then for some $\bar{b} \in B^w$,

$$f(\bar{b}) \neq g(\bar{b}).$$

Let $b = f(\bar{b})$ and $b' = g(\bar{b})$ then $b \neq b'$. Since T_s is Hausdorff there exist disjoint neighbourhoods $U_b \in Nbd_B(b)$ and $U_{b'} \in Nbd_B(b')$. Since f and g are continuous then $f^{-1}(U_b)$ and $g^{-1}(U_{b'})$ are open. Then $f^{-1}(U_b)$ contains a non-empty basic open set $U \subseteq B^w$ and $g^{-1}(U_{b'})$ contains a non-empty basic open set $U' \subseteq B^w$. Since A_s is dense in T_s for each $s \in S$ then

$$U \cap A^w \neq \emptyset, \qquad U' \cap A^w \neq \emptyset.$$

Let $\bar{a} \in U \cap A^w$ and $\overline{a'} \in U' \cap A^w$. Then

$$f|_A(\bar{a}) \in U_b \cap A_s, \qquad g|_A(\overline{a'}) \in U_{b'} \cap A_s.$$

But U_b and $U_{b'}$ are disjoint so

$$f|_A(\bar{a}) \neq g|_A(\overline{a'}).$$

Hence $f|_A \neq g|_A$. $\square$

Corollary 3.3. *Let C_1 and C_2 be Σ algebras. If (B, T) is Hausdorff and C_1 and C_2 are closed continuous extensions of A with respect to (B, T) then $C_1 = C_2$.*

Proof. Immediate from Proposition 3.2. $\square$

Corollary 3.3 motivates the following special case of a closed continuous extension that will be studied in the sequel.

Definition 3.4. Let Σ be an S-sorted signature, let A be a Σ algebra and let (B, T) be an S-indexed family of Hausdorff spaces such that $A \subseteq B$. The unique closed continuous extension of A with respect to (B, T), if it exists, is termed the *Hausdorff extension* of A. We let $H(A)$ denote the Hausdorff extension of A.

In the next two subsections we concern ourselves first with the question of construction and then with the question of existence of Hausdorff extensions.

3.3. Algebraic construction of Hausdorff extensions

Given an S-sorted signature Σ, a Σ algebra A and an S-indexed family (B, T) of Hausdorff spaces such that $A \subseteq B$ we consider the following

Construction Problem. *Assuming that the Hausdorff extension $H(A)$ of A exists, give an explicit algebraic construction of $H(A)$ from A and (B, T).*

The construction problem for $H(A)$ is fundamental since an explicit algebraic construction is required to determine the validity or satisfiability of logical formulas with respect to $H(A)$. In particular, for applications to algebraic specification, if A is a model of some axiomatic specification Φ then we wish to know whether $H(A)$ is also a model of Φ.

We shall present a general solution to the construction problem for Hausdorff extensions. Our construction is applicable to any many-sorted algebra A and any family (B, T) of Hausdorff spaces. It generalises the well known completion of the ring of rational numbers to the ring of real numbers using Cauchy sequences. Furthermore, this construction can be used to identify conditions on A and (B, T) which are sufficient to ensure that $H(A)$ exists. Thus the results of this section will be applied in the next subsection where we consider the existence problem for Hausdorff extensions.

To give a general solution to the construction problem we may assume that the Hausdorff extension $H(A)$ of A exists. We then begin by considering the algebraic relationships that must hold between A and $H(A)$.

First recall the basic topological concepts of nets and their convergence.

Definition 3.5. Let $(I, \leqslant)$ be any directed poset. An element $a \in A_s^I$ from the direct power A^I is termed a *net*. For any $s \in S$ and $a \in A_s^I$ and $b \in B_s$ we say that *a has limit b* or *a converges to b* if, and only if, for each neighbourhood $U \in Nbd_B(b)$ there exists $i \in I$ such that for all $j \geqslant i$,

$$a(j) \in U.$$

By the definition of topological closure, if a converges to b then $b \in Cl(A_s)$.

We say that a is *convergent* if, and only if, for some $b \in B_s$, a has limit b. We say that a is *uniquely convergent* if, and only if, for any b, $b' \in B_s$ if a converges to b and a converges to b' then $b = b'$.

We define the S-indexed family $A^* = A^{*(I)}$ of sets of convergent nets by

$$A_s^* = \{a \in A_s^I \mid a \textit{ is convergent}\}$$

for each sort $s \in S$.

A well known property of the Hausdorff separation axiom is that this axiom is necessary and sufficient to ensure uniqueness of convergence for all nets. Thus we have:

Lemma 3.6. *For any direct poset $(I, \leqslant)$ and any $s \in S$, each element $a \in A_s^*$ is uniquely convergent.*

Proof. See for example [7]. $\square$

Let us consider a fixed but arbitrarily chosen directed poset $(I, \leqslant)$. Now by Lemma 3.6 we can uniquely map every convergent net $a \in A_s^*$ to its limit, which is an element of $Cl(A_s)$, as follows.

Definition 3.7. *Define the S-indexed family $lim : A^* \to Cl(A)$ of limit mappings by*

$$lim_s(a) = b \iff a \text{ converges to } b$$

for each $s \in S$ and each $a \in A_s^*$ and $b \in Cl(A_s)$. By Lemma 3.6, *lim* is well defined.

The family *lim* induces an S-indexed family $\equiv^{lim}$ of equivalence relations on A^* defined by

$$a \equiv_s^{lim} a' \iff lim_s(a) = lim_s(a')$$

for each $s \in S$ and $a, a' \in A_s^*$. $\square$

The limit mappings on A^* have the following basic property.

Lemma 3.8. *For any $w = s(1) \ldots s(n) \in S^+$, $s \in S$ any function symbol $f \in \Sigma_{w,s}$, and for any convergent nets $a_j \in A_{s(j)}^*$ for $1 \leqslant j \leqslant n$,*

$$f_{A^I}(a_1, \ldots, a_n) \text{ converges to } f_{H(A)}(lim_{s(1)}(a_1), \ldots, lim_{s(n)}(a_n)).$$

Proof. Since $H(A)$ is a closed continuous extension of A then $f_{H(A)}$ is continuous. Consider any convergent nets $a_j \in A_{s(j)}^*$ for $1 \leqslant j \leqslant n$ and let $b = f_{H(A)}(lim_{s(1)}(a_1), \ldots, lim_{s(n)}(a_n))$. Consider any neighbourhood $U \in Nbd_{Cl(A)}(b)$. Since $f_{H(A)}$ is continuous, $f_{H(A)}^{-1}(U) \in T_{Cl(A)}^w$ is open in the subspace topology $T_{Cl(A)}$ on $Cl(A)$. Since U is a neighbourhood of b then $(lim_{s(1)}(a_1), \ldots, lim_{s(n)}(a_n)) \in f_{H(A)}^{-1}(U)$. So for some basic open $U' \subseteq f_{H(A)}^{-1}(U)$ in the subspace topology where $U' = U_1' \times \cdots \times U_n'$

$$(lim_{s(1)}(a_1), \ldots, lim_{s(n)}(a_n)) \in U'.$$

Now for each $1 \leqslant j \leqslant n$, a_j is convergent, so there exists $k_j \in I$ such that for all $i \geqslant k_j$,

$$a_j(i) \in U_j'.$$

Let k be an upper bound of $k_1, \ldots, k_n$ then for all $i \geqslant k$

$$(a_1(i), \ldots, a_n(i)) \in U'.$$

Thus for all $i \geqslant k$,

$$f_{A^I}(a_1, \ldots, a_n)(i) = f_A(a_1(i), \ldots, a_n(i))$$
$$= f_{H(A)}(a_1(i), \ldots, a_n(i)) \in U.$$

Since U was arbitrarily chosen then $f_{A^I}(a_1, \ldots, a_n)$ converges to

$$f_{H(A)}(lim_{s(1)}(a_1), \ldots, lim_{s(n)}(a_n)). \qquad \square$$

Lemma 3.8 leads to the following relationships between algebras. First, the S-indexed family A^* of sets of convergent nets is closed under the operations of the direct power A^I. So we have

Corollary 3.9. *The family A^* is the carrier family of a subalgebra of A^I and the family $\delta : A \to A^I$ of diagonal maps is a Σ embedding of A in A^*.*

Proof. We need only show that the family of sets A^* contains each constant of A^I and is closed under the operations of A^I.

Consider any $s \in S$ and constant symbol $c \in \Sigma_{\lambda,s}$. Then c_{A^I} is convergent with limit c_A and thus $c_{A^I} \in A^*_s$.

Consider any $w = s(1) \ldots s(n) \in S^+$, any $s \in S$, any function symbol $f \in \Sigma_{w,s}$ and any convergent nets $a_j \in A^*_{s(j)}$ for $1 \leqslant j \leqslant n$. By Lemma 3.8,

$$f_{A^I}(a_1, \ldots, a_n) \in A^*_s,$$

i.e. A^* is closed under f_{A^I}.

To show that $\delta : A \to A^I$ is a Σ embedding of A in A^*, since δ is a Σ embedding of A in A^I and $A^* \leqslant A^I$ by above, it suffices to show that $\delta(A) \subseteq A^*$. So consider any $s \in S$ and $a \in A_s$. For all $i \in I$, $\delta_s(a)(i) = a$ and so $\delta_s(a)$ converges to a. Thus $\delta_s(a) \in A^*_s$. $\square$

Secondly, the family *lim* of limit maps is a homomorphism from the algebra A^* to the Hausdorff extension $H(A)$.

Proposition 3.10. *The S-indexed family $lim : A^* \to Cl(A)$ of limit mappings is a Σ homomorphism from A^* to $H(A)$.*

Proof. *Consider any* $s \in S$ and constant symbol $c \in \Sigma_{\lambda,s}$. Then

$$lim_s(c_{A^*}) = lim_s(c_{A^I}) \qquad \text{since } A^* \leqslant A^I,$$
$$= c_A = c_{H(A)} \qquad \text{since } A \leqslant H(A).$$

Consider any $w = s(1) \ldots s(n) \in S^+$, $s \in S$, any function symbol $f \in \Sigma_{w,s}$, any convergent $a_j \in A^*_{s(j)}$ for $1 \leqslant j \leqslant n$ and the operation f_{A^*}. Then

$$lim_s(f_{A^*}(a_1, \ldots, a_n)) = lim_s(f_{A^I}(a_1, \ldots, a_n))$$

since $A^* \leqslant A^I$

$$= f_{H(A)}(lim_{s(1)}(a_1), \ldots, lim_{s(n)}(a_n))$$

by Lemma 3.8. $\square$

Thus we may form the homomorphic image $lim(A^*)$ of A^* which stands in the following relationship to A and the Hausdorff extension $H(A)$.

Theorem 3.11. (*Continuous Extension*).

$$A \leqslant lim(A^*) \leqslant H(A).$$

Proof. Since $lim: A^* \to H(A)$ is a Σ homomorphism by Proposition 3.10 then

$$lim(A^*) \leqslant H(A).$$

To show that $A \leqslant lim(A^*)$ note that $A \leqslant H(A)$ by definition of $H(A)$ and $lim(A^*) \leqslant H(A)$ by above. So it suffices to show that $A \subseteq lim(A^*)$. Consider any $s \in S$ and $a \in A_s$ and the diagonal map $\delta_s: A_s \to A_s^I$. By Corollary 3.9, $\delta_s(a) \in A_s^*$ and so $lim_s(\delta_s(a)) \in lim(A^*)_s$. But $lim_s(\delta_s(a)) = a$ and so $a \in lim(A^*)_s$. $\square$

Now since $lim(A^*) \leqslant H(A)$ we may naturally ask: when is $lim(A^*) = H(A)$? In this case we have an algebraic construction of the Hausdorff extension $H(A)$ as required. For a particular adherent element $b \in Cl(A_s)$, if a net converging to b can be constructed over the directed poset $(I, \leqslant)$ then $b \in lim(A^*)_s$. Recall that the set $Nbd_B(b)$ of all neighbourhoods of b in T_s can be partially ordered by $U \succeq U' \Leftrightarrow U \subseteq U'$. Then $(Nbd_B(b), \succeq)$ is a directed poset termed the *neighbourhood poset* of b. A sufficient condition on $(I, \leqslant)$ such that $b \in lim(A^*)_s$ can be given in terms of the neighbourhood poset $(Nbd_B(b), \succeq)$. Recall that if $(I, \leqslant)$ and $(I', \leqslant)$ are posets then an *embedding retraction pair* of monotonic maps is a pair

$$e: (I, \leqslant) \to (I', \leqslant'), \qquad r: (I', \leqslant') \to (I, \leqslant)$$

satisfying

$$r(e(i)) = i \tag{i}$$

for all $i \in I$, and

$$i \leqslant j \;\Rightarrow\; e(i) \leqslant e(j), \qquad i' \leqslant j' \;\Rightarrow\; r(i') \leqslant r(j') \tag{ii}$$

for all $i, j \in I$ and $i', j' \in I'$.

Lemma 3.12. (Adherence). *For any $s \in S$ and $b \in Cl(A_s)$, if there exists an embedding retraction pair of monotonic maps*

$$e: (Nbd_B(b), \preceq) \to (I, \leqslant)$$

$$r: (I, \leqslant) \to (Nbd_B(b), \preceq),$$

then $b \in lim(A^*)_s$.

Proof. Let e and r be an embedding retraction pair of monotonic maps. We must show that for some $a \in A_s^*$, $lim_s(a) = b$. Define a by

$$a(i) = a_{r(i)}$$

for each $i \in I$, where $a_{r(i)} \in r(i) \cap A_s$. Since $b \in Cl(A_s)$ then $a_{r(i)}$ exists for each $i \in I$. Now $lim_s(a) = b$, for consider any $U \in Nbd_B(b)$ and $e(U) \in I$ and any $j \geqslant e(U)$. Since r is monotonic and r and e form an embedding retraction pair then $r(j) \succeq r(e(U)) = U$. By definition

$$a(j) \in r(j) \cap A_s.$$

But $r(j) \succeq U$ so $r(j) \subseteq U$ and so $a(j) \in U$. Since j was arbitrarily chosen then for all $j \geqslant e(U)$, $a(j) \in U$. Since U was arbitrarily chosen then a converges to b. So $a \in A_s^*$ and $lim_s(a) = b$.

Lemma 3.12 suggests that we construct a single directed poset in which the neighbourhood poset of every adherent point can be embedded.

Definition 3.13. Let $\amalg_{s \in S} Cl(A_s)$ denote the disjoint sum (coproduct) of the sets $Cl(A_s)$. Define the poset

$$(I(A), \leqslant^A) = \prod_{(s,b) \in \amalg_{s \in S} Cl(A_s)} (Nbd_B(b), \preceq).$$

Clearly the direct product of a family of directed posets is again directed.

Thus $(I(A), \leqslant^A)$ is the direct product of the neighbourhood posets of all elements of all sets $Cl(A_s)$. Then $(I(A), \leqslant^A)$ has the following strong embedding property with respect to its coordinate posets.

Proposition 3.14. *For any $n \geqslant 1$, any $s(1), \ldots, s(n) \in S$ and any $b_j \in Cl(A_{s(j)})$ for $1 \leqslant j \leqslant n$ there exists an embedding retraction pair of monotonic maps*

$$e : (Nbd_B(b_1), \preceq) \times \cdots \times (Nbd_B(b_n), \preceq) \to (I(A), \leqslant^A),$$

$$r : (I(A), \leqslant^A) \to (Nbd_B(b_1), \preceq) \times \cdots \times (Nbd_B(b_n), \preceq).$$

Proof. Define the embedding e by

$$e(U_1, \ldots, U_n)(s,b) = \begin{cases} U_j, & \text{if } (s,b) = (s(j), b_j); \\ B_s, & \text{otherwise,} \end{cases}$$

for any $U_j \in Nbd_B(b_j)$ for $1 \leqslant j \leqslant n$, any $s \in S$ and $b \in Cl(A_s)$. Define the retraction r by

$$r(i) = (i(s(1), b_1), \ldots, i(s(n), b_n))$$

for each $i \in I(A)$. Clearly e and r are monotonic, e is injective and for any $U_j \in Nbd_B(b_j)$ for $1 \leqslant j \leqslant n$,

$$r(e(U_1, \ldots, U_n)) = (U_1, \ldots, U_n). \quad \square$$

Thus we have the following explicit algebraic construction of the Hausdorff extension $H(A)$ from A and T.

Theorem 3.15 (Construction). *Let $(I, \leqslant) = (I(A), \leqslant^A)$ and let $A^* \leqslant A^{I(A)}$ be the subalgebra of convergent nets. Then $lim(A^*)$ is the Hausdorff extension $H(A)$ of A.*

Proof. Follows immediately from Theorem 3.11, Lemma 3.12 and Proposition 3.14. $\square$

An important property of every Hausdorff extension $H(A)$ is determined by the form of this construction for $H(A)$: the equational theories of A and $H(A)$ are the same.

Theorem 3.16 (Conservative extension). *For any equation $e \in Eqn(\Sigma, X)$,*

$$A \models e \Leftrightarrow H(A) \models e.$$

Proof. By Theorem 3.15, $H(A) = lim(A^*)$ where $I = I(A)$.

($\Rightarrow$) Suppose $A \models e$. Since A^I is a direct power of A then $A^I \models e$. By Corollary 3.9, $A^* \leqslant A^I$ so $A^* \models e$. By Proposition 3.10, $lim(A^*)$ is a homomorphic image of A^* and so $lim(A^*) \models e$. Thus $H(A) \models e$.

($\Leftarrow$) Suppose $H(A) \models e$, then $lim(A^*) \models e$. By Theorem 3.11, $A \leqslant lim(A^*)$ and so $A \models e$. $\square$

Theorem 3.16 is of fundamental importance from the point of view of algebraic specification. If A is a model of an equational specification E (for example an initial model, final model or higher-order initial model) then so is $H(A)$.

It is natural to consider whether our construction of $H(A)$ can be simplified in the presence of stronger topological axioms. In particular, when can we replace an arbitrary directed poset with the usual linear ordering $\leqslant$ on $\mathbf{N}$ and consider convergent sequences rather than arbitrary convergent nets? (Recall for example the Cauchy completion of the ring of rationals to the ring of reals.)

Definition 3.17. Let $(\mathbf{N}, \leqslant)$ be the poset of natural numbers with the usual linear ordering $\leqslant$. A convergent net $a \in A_s^{\mathbf{N}}$ is termed a *convergent sequence.*

Recall the first axiom of countability for a topological space.

Definition 3.18. An S-indexed family (B, T) of topological spaces is 1° *countable* if and only if, for each $s \in S$ the topology T_s is 1° countable, i.e. for each $b \in B_s$ there exists a countable family

$$\langle U_n(b) \in Nbd_B(b) \mid n \in \mathbf{N} \rangle$$

of neighbourhoods of b such that for any neighbourhood $U \in Nbd_B(b)$ of b,

$$U_n(b) \subseteq U$$

for some $n \in \mathbf{N}$.

If (B, T) is 1° countable then we may replace the directed poset $(I(A), \leqslant^A)$ by $(\mathbf{N}, \leqslant^{\mathbf{N}})$ in the construction of the Hausdorff extension $H(A)$.

Theorem 3.19. *Let $(I, \leqslant) = (\mathbf{N}, \leqslant)$. If (B, T) is 1° countable then $H(A) = lim(A^*)$.*

Proof. By Theorem 3.11, $lim(A^*) \leqslant H(A)$ so we need only show that $Cl(A) \subseteq lim(A^*)$. Consider any sort $s \in S$ and any $b \in Cl(A_s)$. Since T_s is 1° countable there exists a countable family

$$\langle U_n(b) \in Nbd_B(b) \mid n \in \mathbf{N} \rangle$$

of neighbourhoods of b such that for any neighbourhood $U \in Nbd_B(b)$, $U_n(b) \subseteq U$ for some $n \in \mathbf{N}$.

Define the family $\langle U'_n(b) \subseteq B_s \mid n \in \mathbf{N} \rangle$ of sets inductively by

$$U'_0(b) = U_0(b)$$

and for any $n \in \mathbf{N}$,

$$U'_{n+1}(b) = U'_n(b) \cap U_{n+1}(b).$$

It is easily shown by induction on n that for all $n \in \mathbf{N}$,

$$U'_n(b) \in Nbd_B(b), \tag{1}$$

$$U'_{n+1}(b) \subseteq U'_n(b), \tag{2}$$

$$U'_n(b) \subseteq U_n(b). \tag{3}$$

We show that for some convergent sequence $a \in A_s^{\mathbf{N}}$, $lim_s(a) = b$. Define a by

$$a(n) = a_n$$

for each $n \in \mathbf{N}$, where $a_n \in U'_n(b) \cap A_s$. Since $b \in Cl(A_s)$ then by (1) above a_n exists for each $n \in \mathbf{N}$. Now consider any neighbourhood $U \in Nbd_B(b)$. For some $n \in \mathbf{N}$, $U_n(b) \subseteq U$ so for such n by (2) and (3) above, $U'_m(b) \subseteq U$ for all $m \geqslant n$. Thus $a(m) \in U$ for all $m \geqslant n$. So a converges to b, i.e. $lim_s(a) = b$. $\square$

3.4. Existence of Hausdorff extensions

In the previous subsection we assumed the existence of the Hausdorff extension $H(A)$ of A and used this fact to derive an explicit algebraic construction of this extension. But how can we tell whether $H(A)$ exists in the first place? We shall consider the following

Existence Problem. *Given an S-sorted signature Σ, a Σ algebra A and an S-indexed family (B, T) of Hausdorff spaces such that $A \subseteq B$, find sufficient conditions on A and (B, T) such that the Hausdorff extension $H(A)$ exists.*

Our approach to this problem will be to analyse the construction of Section 3.3 and determine general conditions sufficient to allow this construction to be carried out. We note however that given more detailed information about the algebraic structure of A and the nature of T we may be able to derive much stronger results than those presented in this section. For example it is a non-trivial result of topological group theory (see for example [19]) that if G is a group and (B, T) is a Hausdorff space such that $G \subseteq B$ and G is continuous in the subspace topology on T then there exists a group $\hat{G}$ which is a closed continuous extension of G in (B, T). The group $\hat{G}$ is usually termed the *bilinear completion of* G rather than the Hausdorff extension. In Section 4 we provide a further example of this type of result by considering necessary and sufficient conditions for the existence of Hausdorff extensions of *second-order algebras.*

We begin with the following elementary fact, which is based on simply checking the feasibility of each step of the construction introduced in Section 3.3.

Proposition 3.20. *Let $(I, \leqslant) = (I(A), \leqslant^A)$. If:*
(i) *A^* is a subalgebra of A^I, and*
(ii) *$\equiv^{lim}$ is a congruence of A^*, and*
(iii) *$A^*/\equiv^{lim}$ is continuous with respect to the induced topology $T_{A^*/\equiv^{lim}}$*
then the Hausdorff extension $H(A)$ of A exists and

$$H(A) \cong A^*/\equiv^{lim} .$$

Proof. Assume (i)–(iii) hold. Define the S-indexed family $\psi : A^*/\equiv^{lim} \to B$ of mappings by

$$\psi_s(a/\equiv^{lim}) = lim_s(a)$$

for each sort $s \in S$ and each $a \in A_s^*$. Now define the Σ algebra $H(A)$ as follows. For each $s \in S$, define

$$H(A)_s = \psi_s((A^*/\equiv^{lim})_s).$$

For each $s \in S$ and constant symbol $c \in \Sigma_{\lambda,s}$, define

$$c_{H(A)} = lim_s(c_{A^*}).$$

For any $w = s(1) \ldots s(n) \in S^+$, any $s \in S$, any $f \in \Sigma_{w,s}$ and any $a_j \in A^*_{s(j)}$ for $1 \leqslant j \leqslant n$, define

$$f_{H(A)}(\psi_{s(1)}(a_1/\equiv^{lim}), \ldots, \psi_{s(n)}(a_n/\equiv^{lim})) = \psi_s(f_{A^*}(a_1, \ldots, a_n)/\equiv^{lim}).$$

Clearly $H(A)$ is well defined as a Σ algebra and $\psi : A^*/\equiv^{lim} \to H(A)$ is a Σ isomorphism. We show that $H(A)$ is a closed continuous extension of A. Recall Definition 3.1.

(i) To show that $H(A)$ extends A consider any sort $s \in S$ and any constant symbol $c \in \Sigma_{\lambda,s}$. Then

$$c_{H(A)} = lim_s(c_{A^*}) = c_A.$$

Consider any $w = s(1) \ldots s(n) \in S^+$, any sort $s \in S$, any function symbol $f \in \Sigma_{w,s}$ and any $a_j \in A_{s(j)}$ for $1 \leqslant j \leqslant n$. Then

$$\begin{aligned}
&f_{H(A)}(a_1, \ldots, a_n)\\
&\quad = f_{H(A)}(\psi_{s(1)}(\delta_{s(1)}(a_1)/\equiv^{lim}), \ldots, \psi_{s(n)}(\delta_{s(n)}(a_n)/\equiv^{lim}))\\
&\quad = \psi_s(f_{A^*}(\delta_{s(1)}(a_1), \ldots, \delta_{s(n)}(a_n))/\equiv^{lim}) \quad \text{by definition of } f_{H(A)}\\
&\quad = lim_s(f_{A^*}(\delta_{s(1)}(a_1), \ldots, \delta_{s(n)}(a_n))) \quad \text{by definition of } \psi\\
&\quad = lim_s(\delta_s(f_A(a_1, \ldots, a_n))) \quad \text{by Corollary 3.9}\\
&\quad = f_A(a_1, \ldots, a_n).
\end{aligned}$$

Thus $H(A)$ is an extension of A.

(ii) By the definition of ψ, for any sort $s \in S$,

$$H(A)_s = \psi_s((A^*/\equiv^{lim})_s) = lim(A^*)_s.$$

Since $I = I(A)$ then by Lemma 3.12 and Proposition 3.14, $H(A)_s = Cl(A_s)$.

(iii) For each sort $s \in S$ let $T_s^{H(A)}$ be the subspace topology on $H(A)_s$ and define the induced topology $T_s^{A^*/\equiv^{lim}}$ on $(A^*/\equiv^{lim})_s$ by

$$U \in T^{A^*/\equiv^{lim}} \Leftrightarrow \psi_s(U) \in T_s^{H(A)}$$

for each $U \subseteq (A^*/\equiv^{lim})_s$. Then

$$\psi : (A^*/\equiv^{lim}, T^{A^*/\equiv^{lim}}) \to (H(A), T^{H(A)})$$

is a homomorphism. Since ψ is both a homomorphism and a Σ isomorphism and $A^*/\equiv^{lim}$ is continuous then so is $H(A)$.

So by (i)–(iii), $H(A)$ is a closed continuous extension of A. $\square$

Condition (iii) of Proposition 3.20 is not very satisfactory since it indicates that even if A is continuous with respect to the subspace topology on T and the algebra $A^*/\equiv^{lim}$ is well defined, the latter may fail to be continuous. However, if we assume slightly stronger separation properties on T then this possibility cannot arise. Recall the separation axiom of regularity.

Definition 3.21. An S-indexed family (B, T) of topological spaces is *regular* if, and only if, for each sort $s \in S$ the topology T_s is regular, i.e. for any $b \in B_s$ and closed set $U \in T_s$ not containing b there exist open sets $V, V' \in T_s$ with $b \in V$, $U \subseteq V'$ and

$$V \cap V' = \emptyset .$$

Clearly every regular space is Hausdorff. Regular spaces include for example all metric spaces. Recall that if $T_{s(1)}, \ldots, T_{s(n)}$ are regular topologies on the sets $B_{s(1)}, \ldots, B_{s(n)}$ then the product topology T^w on the cartesian product A^w is regular. Regular spaces have the following important property.

Theorem 3.22 *Let (X, T) be a topological space, let (Y, T') be a regular space, let $D \subseteq X$ be dense in T and let $f: D \to Y$ be a continuous map. Then f has a continuous extension $\hat{f}: X \to Y$ if, and only if, the filterbase $f(D \cap Nbd_X(x))$ converges for each $x \in X$. If $\hat{f}$ exists then it is unique.*

Proof. See for example [4]. $\square$

To apply Theorem 3.22 to Proposition 3.20 we relate convergence on nets to convergence on filterbases.

Lemma 3.23. *Let $(I, \leqslant) = (I(A), \leqslant^A)$ and suppose A is dense in T. For any $w = s(1) \ldots s(n) \in S^+$, any $s \in S$, any operation symbol $f \in \Sigma_{w,s}$, any $b_j \in B_{s(j)}$ for $1 \leqslant j \leqslant n$, and any $b \in B_s$, if*

$$f_{A^I}(a_1, \ldots, a_n) \text{ converges to } b$$

for all $a_j \in A^I_{s(j)}$ converging to b_j for $1 \leqslant j \leqslant n$ then

$$f_A(A^w \cap Nbd_B(b_1, \ldots, b_n)) \text{ converges to } b .$$

Proof. We prove the contrapositive. Suppose $f_A(A^w \cap Nbd_B(b_1, \ldots, b_n))$ does not converge to b. Then for some $V \in Nbd_B(b)$ there is no $U \in Nbd_B(b_1, \ldots, b_n)$ such that

$$f_A(A^w \cap U) \subseteq V .$$

For each $U \in Nbd_B(b_1, \ldots, b_n)$ let $(a_1^U, \ldots, a_n^U) \in A^w \cap U$ be an n-tuple satisfying

$$f_A(a_1^U, \ldots, a_n^U) \notin V .$$

By density of A in T, for each $1 \leqslant j \leqslant n$, $b_j \in Cl(A_{s(j)})$, and since $I = I(A)$ then by Proposition 3.14 there exists an embedding retraction pair

$$e: (Nbd_B(b_1), \preceq) \times \cdots \times (Nbd_B(b_n), \preceq) \to (I, \leqslant),$$

$$r: (I, \leqslant) \to (Nbd_B(b_1), \preceq) \times \cdots \times (Nbd_B(b_n), \preceq).$$

For each $1 \leqslant j \leqslant n$, define $a_j \in A^I_{s(j)}$ by

$$a_j(i) = a_j^{r(i)}$$

for each $i \in I$. Consider any $1 \leqslant j \leqslant n$ and $U^j \in Nbd_B(b_j)$ and

$$U = B_{s(1)} \times \cdots \times B_{s(j-1)} \times U^j \times B_{s(j+1)} \times \cdots \times B_{s(n)}.$$

For any $i \geqslant e(U)$ we have $r(i) \succeq r(e(U)) = U$. So $r(i)_j \succeq U^j$ and hence $r(i)_j \subseteq U^j$. But

$$a_j(i) = a_j^{r(i)} \in r(i)_j.$$

So for all $i \geqslant e(U)$, $a_j(i) \in U^j$. Since j and U^j were arbitrarily chosen then a_j converges to b_j for each $1 \leqslant j \leqslant n$. Now consider $f_{A^I}(a_1, \ldots, a_n)$. For all $i \in I$,

$$\begin{aligned} f_{A^I}(a_1, \ldots, a_n)(i) &= f_A(a_1(i), \ldots, a_n(i)) \\ &= f_A(a_1^{r(i)}, \ldots, a_n^{r(i)}) \notin V. \end{aligned}$$

So $f_{A^I}(a_1, \ldots, a_n)$ does not converge to b. $\square$

Corollary 3.24. *Let $(I, \leqslant) = (I(A), \leqslant^A)$ and suppose A is dense in T. If:*

(i) *A^* is a subalgebra of A^I, and*

(ii) *$\equiv^{lim}$ is a congruence on A^**

then for each $w = s(1) \ldots s(n) \in S^+$, each $s \in S$, each operation symbol $f \in \Sigma_{w,s}$ and for any $b_j \in B_{s(j)}$, for $1 \leqslant j \leqslant n$,

$$f_A(A^w \cap Nbd_B(b_1, \ldots, b_n))$$

converges.

Proof. Suppose A is dense in T and assume (i) and (ii) hold. Consider any $w = s(1) \ldots s(n) \in S^+$, any $s \in S$ any operation symbol $f \in \Sigma_{w,s}$ and any $b_j \in B_{s(j)}$ for $1 \leqslant j \leqslant n$. Since A is dense in T then $b_j \in Cl(A_{s(j)})$ for each $1 \leqslant j \leqslant n$. Since $I = I(A)$, by Lemma 3.12 and Proposition 3.14, for each $1 \leqslant j \leqslant n$ there exists $a_j \in A^*_{s(j)}$ such that a_j converges to b_j. By (i) for some $b \in B_s$, $f_{A^I}(a_1, \ldots, a_n)$ converges to b.

Now consider any other $a'_j \in A^*_{s(j)}$ converging to b_j for $1 \leqslant j \leqslant n$. Then $a_j \equiv^{lim}_{s(j)} a'_j$ for $1 \leqslant j \leqslant n$ and so by (ii)

$$f_{A^I}(a_1, \ldots, a_n) \equiv^{lim}_s f_{A^I}(a'_1, \ldots, a'_n).$$

Thus $f_{A^I}(a'_1, \ldots, a'_n)$ converges to b.

Therefore for all $a_j \in A^I_{s(j)}$ converging to b_j, for $1 \leqslant j \leqslant n$, $f_{A^I}(a_1, \ldots, a_n)$ converges to b. So by Lemma 3.2.3,

$$f_A(A^w \cap Nbd_B(b_1, \ldots, b_n))$$

converges to b. $\square$

Combining Corollary 3.24 and Theorem 3.22 we can refine Proposition 3.20 to the following.

Theorem 3.25. *Let $(I, \leqslant) = (I(A), \leqslant^A)$ and suppose (B, T) is a family of regular spaces. If*:

(i) *A is continuous with respect to the subspace topology, and*
(ii) *A^* is a subalgebra of A^I, and*
(iii) *$\equiv^{lim}$ is a congruence on A^**

then the Hausdorff extension $H(A)$ of A exists and

$$H(A) \cong A^* / \equiv^{lim}.$$

Proof. Suppose (B, T) is regular and assume (i)–(iii) hold. Consider any $w = s(1) \ldots s(n) \in S^+$, any $s \in S$ and any function symbol $f \in \Sigma_{w,s}$. By (ii), (iii) and Corollary 3.24 for any $b_j \in Cl(A_{s(j)})$ for $1 \leqslant j \leqslant n$ the filterbase $f_A(A^w \cap Nbd_B(b_1, \ldots, b_n))$ converges. By (i) f_A is continuous so by Theorem 3.22, f_A has an extension

$$\hat{f} : Cl(A_{s(1)}) \times \cdots \times Cl(A_{s(n)}) \to Cl(A_s)$$

which is continuous with respect to the subspace topology.

We construct the Hausdorff extension $H(A)$ of A as follows. For each sort $s \in S$ define the carrier set $H(A)_s = Cl(A_s)$. For each $s \in S$ and constant symbol $c \in \Sigma_{\lambda,s}$ define

$$c_{H(A)} = c_A.$$

For each $w \in S^+$, $s \in S$ and operation symbol $f \in \Sigma_{w,s}$ define

$$f_{H(A)} = \hat{f}.$$

Clearly $H(A)$ is the Hausdorff extension of A.

Since $I = I(A)$, by Theorem 3.15, $lim(A^*)_s = Cl(A_s) = H(A)_s$ for each $s \in S$. So by Proposition 3.10, $lim : A \to H(A)$ is an epimorphism with kernel $\equiv^{lim}$. Then by the First Homomorphism Theorem $A^* / \equiv^{lim} \cong lim(A^*) \cong H(A)$. $\square$

4. Hausdorff extensions of second-order algebras

In this section we consider, as an example which illustrates the techniques introduced in Section 3, Hausdorff extensions of second-order algebras. We will characterise necessary and sufficient conditions under which a second-order algebra A has a Hausdorff extension with respect to a natural topology T known as the *Tychonoff topology*. For this characterisation we introduce the class of *stable functions* on the carriers of A. The algebra A is stable if, and only if, each operation of A is stable. Our main result in this section is the Extension Theorem 4.13 which establishes that if A is stable then A has a Hausdorff extension. Furthermore, if A is dense in the Tychonoff topology T then A has a Hausdorff extension if, and only if, A is stable.

This example provides further evidence of the fact that the existence results for Hausdorff extensions, considered in Section 3.4, can be substantially strengthened by using detailed information about the algebra A and the topological space (B, T).

We begin by recalling the concepts of a second-order type structure, second-order signature and second-order algebra. A systematic account of higher-order algebra is [10].

Definition 4.1. Let B be any non-empty set, the members of which will be termed *basic types*, the set B being termed a *type basis*. The first two levels $H_0(B)$ and $H_1(B)$ in the *type hierarchy* generated by B are defined by,

$$H_0(B) = B$$

and

$$H_1(B) = H_0(B) \cup \{(\sigma \to \tau) \mid \sigma, \tau \in H_0(B)\}.$$

Each non-basic type $(\sigma \to \tau) \in H_1(B)$ is termed a *function type* or *arrow type*. Each basic type $\tau \in B$ has order 0 and each function type $(\sigma \to \tau) \in H_1(B)$ has order 1. A *second-order type structure* S over a type basis B is a subset $S \subseteq H_1(B)$ which is closed under subtypes in the sense that for any function type $(\sigma \to \tau) \in S$ we have $\sigma, \tau \in S$.

Definition 4.2. Let S be a second-order type structure over a type basis B. An S-sorted *second-order signature* Σ is an S-sorted signature such that for each function type $(\sigma \to \tau) \in S$ we have a binary *evaluation operation symbol*

$$eval^{(\sigma \to \tau)} \in \Sigma_{(\sigma \to \tau)\sigma, \tau}.$$

When the types σ and τ are clear we let *eval* denote the evaluation operation symbol $eval^{(\sigma \to \tau)}$. Next we introduce the intended interpretations of a second-order signature Σ.

Definition 4.3. Let S be a second-order type structure over a type basis B. Let Σ be an S-sorted second-order signature and let A be an S-sorted Σ algebra. We say that A is a *second-order Σ algebra* if, and only if, for each function type $(\sigma \to \tau) \in S$ we have $A_{(\sigma \to \tau)} \subseteq [A_\sigma \to A_\tau]$, i.e. $A_{(\sigma \to \tau)}$ is a subset of the set of all (total) functions from A_σ to A_τ. Furthermore, for each function type $(\sigma \to \tau) \in S$, $eval_A^{(\sigma \to \tau)} : A_{(\sigma \to \tau)} \times A_\sigma \to A_\tau$ is the *evaluation operation* on the function space $A_{(\sigma \to \tau)}$ defined by

$$eval_A^{(\sigma \to \tau)}(a, b) = a(b)$$

for each $a \in A_{(\sigma \to \tau)}$ and $b \in A_\sigma$.

In the sequel we consider a fixed but arbitrarily chosen second-order type structure S over a type basis B, S-sorted second-order signature Σ, and second-order Σ algebra A.

The algebra A has a natural topology by taking the discrete topology on A_τ for each basic type τ and the product or Tychonoff topology on $A_{(\sigma\to\tau)}$ for each function type $(\sigma \to \tau) \in S$. This family of topologies for A we term the *Tychonoff topology* on A. Continuity of an operation f_A of A with respect to the Tychonoff topology captures a natural computational intuition that determining the value of f_A on any arguments $a_1, \ldots, a_n$ requires only a finite amount of information about each argument a_i.

We define the set-theoretic completion $\hat{A}_\tau$ of each carrier set A_τ of A together with the Tychonoff topology T_τ on $\hat{A}_\tau$ as follows.

Definition 4.4. Define the S-indexed family of sets $\hat{A}$ and the S-indexed family of *Tychonoff topologies* T on $\hat{A}$ by:

(i) For each basic type $\tau \in S$ define $\hat{A}_\tau = A_\tau$ and let $T_\tau = \wp(A_\tau)$ be the discrete topology on $\hat{A}_\tau$.

(ii) For each function type $(\sigma \to \tau) \in S$ define $\hat{A}_{(\sigma\to\tau)} = [A_\sigma \to A_\tau]$, i.e. $\hat{A}_{(\sigma\to\tau)}$ is the set of all total functions from A_σ to A_τ. Let $T_{(\sigma\to\tau)}$ be the product or Tychonoff topology on $\hat{A}_{(\sigma\to\tau)}$ with subbasic open sets of the form

$$O_{a,b} = \{a' \in \hat{A}_{(\sigma\to\tau)} \mid a'(b) = a(b)\}$$

for each $a \in \hat{A}_{(\sigma\to\tau)}$ and $b \in \hat{A}_\sigma$.

Let us characterise the continuous mappings $f: \hat{A}^w \to \hat{A}_\tau$ for $w \in S^+$ and $\tau \in S$ with respect to the Tychonoff topology. First consider the case that τ is a basic type.

Proposition 4.5. *Let $w \in S^+$, let $\tau \in B$ and let $f: \hat{A}^w \to \hat{A}_\tau$ be any mapping. Then f is continuous if, and only if, for any $\bar{a} \in \hat{A}^w$ there is a basic open set $U \in T^w$ with $\bar{a} \in U$ and f constant valued on U.*

Proof. Exercise. $\square$

By Proposition 4.5, if $w = \tau(1), \ldots, \tau(n)$ and every $\tau(i)$ is a basic type then $f: \hat{A}^w \to \hat{A}_\tau$ is trivially continuous. However, if one or more of the domain types $\tau(i)$ is a function type, the f is continuous if, and only if, f is *finitely determined* in the sense that for any $(a_1, \ldots, a_n) \in \hat{A}^w$, $f(a_1, \ldots, a_n)$ is determined by just a finite part of each function argument a_i.

Let us characterise continuity in the case that τ is a function type.

Proposition 4.6. *Let $w \in S^+$, let $(\sigma \to \tau) \in S$ and let $f: \hat{A}^w \to \hat{A}_{(\sigma\to\tau)}$ be any mapping. Then f is continuous if, and only if, its uncurried form*

$$uc(f): \hat{A}^w \times \hat{A}_\sigma \to \hat{A}_\tau$$

given by $uc(f)(a,b) = f(a)(b)$ is continuous.

Proof. Exercise. □

It is easily shown that the evaluation function $eval: \hat{A}_{(\sigma\to\tau)} \times \hat{A}_\sigma \to \hat{A}_\tau$ is continuous. We will establish necessary and sufficient conditions for the existence of the Hausdorff extension $H(A)$ of A with respect to $(\hat{A}, T)$. For this we must characterise those operations of A which have a continuous extension to the closures of the carriers $Cl(A)$ of A in $(\hat{A}, T)$. We characterise such operations by introducing the class of *stable functions* over subsets of A.

Definition 4.7. Let $B \subseteq \hat{A}$ be any S-indexed family of subsets. For any $w \in S^+$ and $\tau \in S$ we define the class of *stable* functions on B

$$f: B^w \to B_\tau$$

as follows.

(i) For each basic type $\tau \in S$ and any $f: B^w \to B_\tau$, f is stable if, and only if, for any $\bar{a} \in \hat{A}^w$ there exists a neighbourhood $U \in Nbd_A(\bar{a})$ such that f is constant valued on $U \cap B^w$.

(ii) For each function type $(\sigma \to \tau) \in S$ and any $f: B^w \to B_{(\sigma\to\tau)}$, f is stable if, and only if, its uncurried form $uc(f): B^w \times B_\sigma \to B_\tau$ is stable.

Notice that in Definition 4.7, if $B = \hat{A}$ then the stable functions on B are precisely the continuous functions on B. In general this is not the case as we will show. The following lemma shows that examples of stable functions can arise as restrictions of continuous functions on $\hat{A}$.

Lemma 4.8. *Let $B \subseteq \hat{A}$. For any $w \in S^+$, $\tau \in S$ and any continuous map $f: \hat{A}^w \to \hat{A}_\tau$ the restriction $f | B: B^w \to \hat{A}_\tau$ is stable.*

Proof. Consider any basic type $\tau \in S$ and continuous map $f: \hat{A}^w \to \hat{A}_\tau$. Consider any $\bar{a} \in \hat{A}^w$ and let $b = f(\bar{a})$. Then $\{b\} \in T_\tau$ is basic open. Since f is continuous then $U = f^{-1}(\{b\})$ is open and clearly $\bar{a} \in U$. Since f is constant valued on U then $f | B$ is constant valued on $U \cap B^w$ and so $f | B$ is stable.

Consider any function type $(\sigma \to \tau) \in S$ and continuous $f: \hat{A}^w \to \hat{A}_{(\sigma\to\tau)}$. By Proposition 4.6, the uncurried form $uc(f): \hat{A}^w \times \hat{A}_\sigma \to \hat{A}_\tau$ is continuous. So by above $uc(f) | B$ is stable. But $uc(f) | B = uc(f | B)$ and hence $f | B$ is stable. □

So for example the evaluation functions of A are stable. Next we show that stability implies continuity, although the converse need not hold.

Lemma 4.9. (i) *Let $B \subseteq \hat{A}$. For any $w \in S^+$, any type $\tau \in S$ and any function $f: B^w \to B_\tau$, if f is stable then f is continuous with respect to the subspace topology induced on $(\hat{A}, T)$.*

(ii) *There exists a second-order algebra C and a continuous function $f: C^w \to C_\tau$ on the subspace topology induced on $(\hat{C}, T)$ which is not stable.*

Proof. (i) Consider any $w \in S^+$. Consider any basic type $\tau \in S$ and stable $f: B^w \to B_\tau$. Consider any $b \in B_\tau$. We show that $f^{-1}(\{b\})$ is open. Consider any $\bar{b} \in f^{-1}(\{b\})$. By stability, for some $U_{\bar{b}} \in T^w$ we have $\bar{b} \in U_{\bar{b}}$ and f is constant valued on $U_{\bar{b}} \cap B^w$. Now $U_{\bar{b}} \cap B^w$ is open in the subspace topology on T^w and $\bar{b} \in U_{\bar{b}} \cap B^w$. Thus

$$f^{-1}(\{b\}) = \bigcup_{\bar{b} \in f^{-1}(\{b\})} U_{\bar{b}} \cap B^w$$

is open in the subspace topology on T^w. So f is continuous.

Consider any function type $(\sigma \to \tau) \in S$ and stable $f: B^w \to B_{(\sigma \to \tau)}$. Then the uncurried form $uc(f): B^w \times B_\sigma \to B_\tau$ is stable. So by above $uc(f)$ is continuous in the subspace topology on T and hence f is also continuous.

(ii) Consider the second-order type structure $S = \{nat, \tau, (nat \to \tau)\}$ over the type basis $\{nat, \tau\}$. Define the S-sorted second-order algebra C as follows. Let $C_{nat} = \mathbf{N}$, $C_\tau = \{1, 2, 3\}$ and $C_{(nat \to \tau)} = [C_{nat} \to C_\tau]/\{\mathbf{1}\}$ where $\mathbf{1}: \mathbf{N} \to \{1, 2, 3\}$ is given by $\mathbf{1}(n) = 1$ for all $n \in \mathbf{N}$.

Besides the evaluation mapping $eval: C_{(nat \to \tau)} \times C_{nat} \to C_\tau$ let C have one unary operation $f: C_{(nat \to \tau)} \to C_\tau$ defined by

$$f(a) = a(n)$$

for any $a \in C_{(nat \to \tau)}$, where n is the least number such that $a(n) \neq 1$. By definition of $C_{(nat \to \tau)}$, f is well defined and clearly f is continuous. However f is not stable for consider $\mathbf{1} \in \hat{C}_{(nat \to \tau)}$ and let T be the Tychonoff topology on $\hat{C}$. Suppose for a contradiction that there exists $U \in T_{(nat \to \tau)}$ such that $\mathbf{1} \in U$ and f is constant valued on $U \cap C_{(nat \to \tau)}$. Then there is a basic $U = \langle U_1, \ldots, U_n \rangle \in T_{(nat \to \tau)}$ such that $\mathbf{1} \in U$ and f is constant valued on $U \cap C_{(nat \to \tau)}$ which is impossible. $\square$

We will show that every stable function on A extends continuously to a stable function on the closures $Cl(A)$ of the carriers of A in the space $(\hat{A}, T)$. For this we require one simple lemma.

Lemma 4.10. *For any $w \in S^+$, any $\tau \in S$ and any function $f: A^w \to A_\tau$ and for any $\bar{a} \in Cl(A)^w$ and neighbourhoods $U, V \in Nbd_{\hat{A}}(\bar{a})$ of $\bar{a}$, if f is constant valued on $U \cap A^w$ and $V \cap A^w$ then*

$$f(\bar{x}) = f(\bar{y})$$

for all $\bar{x} \in U \cap A^w$ and $\bar{y} \in V \cap A^w$.

Proof. Suppose $w = \tau(1) \ldots \tau(n)$. Since U and V are neighbourhoods of $\bar{a}$ then so is $U \cap V$. Now $U \cap V \cap A^w$ is non-empty since $A_{\tau(i)}$ is dense in $Cl(A_{\tau(i)})$ for $1 \leqslant i \leqslant n$. So consider any $\bar{z} \in U \cap V \cap A^w$. For any $\bar{x} \in U \cap A^w$ and $\bar{y} \in V \cap A^w$,

$$f(\bar{x}) = f(\bar{z})$$

since f is constant valued on $U \cap A^w$

$$= f(\bar{y})$$

since f is constant valued on $V \cap A^w$. □

Theorem 4.11. *For any $w \in S^+$, $\tau \in S$ and stable function $f: A^w \to A_\tau$ there exists a unique stable function $\hat{f}: Cl(A)^w \to Cl(A_\tau)$ which extends f.*

Proof. Since T_τ is Hausdorff for each $\tau \in S$, and any stable extension $\hat{f}$ of f is continuous by Lemma 4.9(i), then by Proposition 3.2, $\hat{f}$ must be unique. Thus we need only prove the existence of $\hat{f}$. We define $\hat{f}$ for each stable $f: A^w \to A_\tau$ as follows.

(i) Consider any basic type $\tau \in S$ and stable $f: A^w \to A_\tau$. For any $\bar{a} \in Cl(A)^w$ let $U_{\bar{a}} \in Nbd_{\hat{A}}(\bar{a})$ be a neighbourhood of $\bar{a}$ such that f is constant valued on $U_{\bar{a}} \cap A^w$. Since $A_{\tau(i)}$ is dense in $Cl(A_{\tau(i)})$ for $1 \leqslant i \leqslant n$ then $U_{\bar{a}} \cap A^w \neq \emptyset$. So choosing any $\bar{b} \in U_{\bar{a}} \cap A^w$ define

$$\hat{f}(\bar{a}) = f(\bar{b}).$$

By Lemma 4.10, $\hat{f}(\bar{a})$ does not depend upon the choice of $U_{\bar{a}}$. To show that $\hat{f}$ is stable and extends f, consider any $\bar{a} \in \hat{A}^w$. Since f is stable there exists $U_{\bar{a}} \in Nbd_{\hat{A}}(\bar{a})$ such that f is constant valued on $U_{\bar{a}} \cap A^w$. Consider any $\overline{a'} \in U_{\bar{a}} \cap Cl(A)^w$. By definition

$$\hat{f}(\overline{a'}) = f(\bar{b})$$

for any $\bar{b} \in U_{\bar{a}} \cap A^w$. So $\hat{f}$ is constant valued on $U_{\bar{a}} \cap Cl(A)^w$. Thus $\hat{f}$ is stable. Clearly $\hat{f}$ extends f.

(ii) Consider any function type $(\sigma \to \tau) \in S$ and any stable $f: A^w \to A_{(\sigma \to \tau)}$. By definition the uncurried form $uc(f): A^w \times A_\sigma \to A_\tau$ is stable so define

$$\hat{f}(\bar{a})(b) = \widehat{uc(f)}(\bar{a}, b)$$

for any $\bar{a} \in Cl(A)^w$ and $b \in Cl(A_\sigma)$. To show that $\hat{f}$ is stable and extends f, by above $uc(f)$ is stable. So by (i) above $uc(\hat{f}): Cl(A)^w \times Cl(A_\sigma) \to Cl(A_\tau)$ is stable and extends $uc(f)$. Hence $\hat{f}$ is stable and extends f. □

We can apply the definition of stability to give necessary and sufficient conditions for the existence of a Hausdorff extension $H(A)$ of A.

Definition 4.12. Let S be a second-order type structure, Σ be an S-sorted second-order signature and A be a second-order Σ algebra. We say that A is *stable* if, and only if, for each $w \in S^+$ and $\tau \in S$ and each operation symbol $f \in \Sigma_{w,\tau}$ the operation f_A is stable.

Theorem 4.13 (Extension).
(i) *If A is stable then A has a Hausdorff extension $H(A)$.*
(ii) *If A is dense in T then A has a Hausdorff extension $H(A)$ if, and only if, A is stable.*

Proof. (i) Immediate from Lemma 4.1(i) and Theorem 4.11.

(ii) Suppose A is dense in T. $\Leftarrow$ Immediate from (i). $\Rightarrow$ By density, for each $\tau \in S$, $H(A)_\tau = \hat{A}_\tau$. For any $w \in S^+$, any $\tau \in S$ and any $f \in \Sigma_{w,\tau}$ the operation $f_{H(A)} : \hat{A}^w \to \hat{A}_\tau$ is continuous. So by Lemma 4.8, f_A is stable. Thus A is stable. $\square$

We conclude by illustrating how the method of Hausdorff extension can be used in the specification of an uncountable second-order algebra.

Example 4.14. Our example is adapted from a hardware specification case study in Meinke and Steggles [21]. We consider countable and uncountable second-order algebras of *convolution* over *streams* over the ring of integers. Such algebras have a common second-order signature Σ^{Conv^n} defined as follows.

Define the second-order type structure $S = \{nat, ring, (nat \to ring)\}$ over the type basis $B = \{nat, ring\}$. Define the second-order signature $\Sigma = \Sigma^{Conv^n}$ by

$$\Sigma_{\lambda, nat} = \{0\}, \qquad \Sigma_{\lambda, ring} = \{0, 1\}, \qquad \Sigma_{\lambda, (nat \to ring)} = \{\bar{0}\},$$

$$\Sigma_{nat, nat} = succ, \qquad \Sigma_{ring, ring} = \{\ -\ \},$$

$$\Sigma_{ring^2, ring} = \{+, \times\}, \qquad \Sigma_{(nat \to ring) nat, ring} = \{eval\},$$

$$\Sigma_{ring (nat \to ring), (nat \to ring)} = \{append\}, \qquad \Sigma_{ring^n (nat \to ring), (nat \to ring)} = \{conv^n\}.$$

The signature Σ^{Conv^n} has a standard minimal extensional model, namely the (countable) second-order algebra A of convolution over almost everywhere zero streams. This algebra has following carrier sets

$$A_{nat} = \mathbf{N}, \qquad A_{ring} = \mathbf{Z},$$

$$A_{(nat \to ring)} = \{\alpha : \mathbf{N} \to \mathbf{Z} \mid \alpha(n) = 0 \text{ for all but finitely many } n \in \mathbf{N}\}.$$

The constants and operations of A are defined as follows.

$$0_A = 0, \qquad succ_A(n) = n + 1,$$

$$0_A = 0, \qquad 1_A = 1,$$

$$-_A(x) = -x, \qquad +_A(x, y) = x + y, \qquad \times_A(x, y) = xy,$$

$$\bar{0}_A(n) = 0, \qquad eval_A(\alpha, n) = \alpha(n),$$

$$append_A(x, \alpha)(0) = x, \qquad append_A(x, \alpha)(n + 1) = \alpha(n),$$

$$conv_A(x_1, \ldots, x_n, \alpha)(m) = x_1\alpha(m) + \cdots + x_n\alpha(m + n).$$

Clearly A is a countable algebra.

Now Σ^{Conv^n} also has a standard uncountable interpretation, namely the algebra of convolution over *all* streams of integers. This algebra may be obtained as the Hausdorff extension of A as follows. Let

$$\hat{A} = \langle \hat{A}_\tau \mid \tau = nat, ring, (nat \to ring) \rangle$$

be the family of completions of the carriers of A and let

$$T = \langle T_\tau \mid \tau = nat, ring, (nat \to ring) \rangle$$

be the Tychonoff topology for $\hat{A}$ as given by Definition 4.4. It is easily shown that $conv^n_A$ is a stable operation. (All other operations of A are trivially stable.) So by Theorem 4.13, A has a Hausdorff extension $H(A)$ with respect to $(\hat{A}, T)$. Furthermore $A_{(nat \to ring)}$ is dense in $T_{(nat \to ring)}$ and so

$$H(A)_{(nat \to ring)} = [\mathbf{N} \to \mathbf{Z}].$$

Thus $H(A)$ is an uncountable algebra. We leave it to the reader, as an easy exercise, to calculate the definition of all operations in the algebra $H(A)$, in particular to confirm that $conv^n_{H(A)}$ is the convolution operation extended to *all* streams of integers. (Notice that T is 1° countable and so the simplified construction of the Hausdorff extension of Theorem 3.19 can be applied.)

It is beyond the scope of this paper to consider higher-order initial algebra specifications of the dense subalgebra A of $H(A)$. Let us simply observe the following

Fact. *A has a finite equational specification E^{Conv^n} under second-order initial algebra semantics.*

Proof. See Meinke and Steggles [21]. □

By Theorem 3.16, $H(A) \models E^{Conv^n}$. Thus E^{Conv^n} provides a finite equational specification of the uncountable algebra $H(A)$ of convolution on streams of integers under second-order initial semantics followed by Hausdorff extension.

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References

[1] A.V. Arkhangel'skii, Classes of topological groups, *Uspekhi. Mat. Nauk.* **36** (1981) 127–146.
[2] G. Birkhoff, On the structure of abstract algebras, *Proc. Camb. Philos. Soc.* **31** (1935) 433–454.

[3] P.M. Cohn, *Universal Algebra* (D. Reidel, Dordrecht, 2nd ed., 1981).
[4] J. Dugundji, *Topology* (Allyn and Bacon, Boston, 1966).
[5] H. Ehrig et al., H. Ehrig, F. Parisi-Presicce, P. Boehm, C. Rieckhoff, C. Dimitrovici and M. Grosse-Rhode, Algebraic data type and process specifications based on projection spaces, in: D. Sanella and A. Tarlecki, eds., *Recent Trends in Data Type Specification*, Lecture Notes in Computer Science, Vol. 332 (Springer, Berlin, 1988) 23–43.
[6] J.A. Goguen, J. Thatcher, E. Wagner and J.B. Wright, Abstract data types as initial algebras and the correctness of data representations, in: A. Klinger, ed., *Computer Graphics, Pattern Recognition and Data Structure* (IEEE, 1975) 89–93.
[7] J.L. Kelley, *General Topology* (Van Nostrand, Princeton, 1955).
[8] A.I. Mal'cev, Free topological algebras, *Izv. Acad. Nauk. SSSR* **21** (1957) 171–198. English translation: *Trans. Amer. Math. Soc.* **17** (1961).
[9] A.A. Markov, On free topological groups, *Izv. Acad. Nauk. SSSR* **9** (1945) 3–64. English translation: *Trans Amer. Math. Soc.* **8** (1962) 195–272.
[10] K. Meinke, Universal algebra in higher types, *Theoret. Comput. Sci.* **100** (1992) 385–417.
[11] K. Meinke and J.V. Tucker, Universal algebra, in: S. Abramsky, D. Gabbay and T.S.E. Maibaum, eds., *Handbook of Logic in Computer Science* (Oxford University Press, Oxford, 1993) 189–411.
[12] B. Möller, Higher-order algebraic specification, Habilitationsschrift, Fakultät Mathematik und Informatik, TU München, 1987.
[13] L.S. Pontrjagin, *Topological Groups* (Princeton University Press, Princeton, 2nd ed., 1954).
[14] M. Smyth, Topology, in: S. Abramsky, D. Gabbay and T.S.E. Maibaum, eds., *Handbook of Logic in Computer Science* (Oxford University Press, Oxford, 1993) 642–761.
[15] V. Stoltenberg-Hansen and J.V. Tucker, Algebraic equations and fixed-point equations in inverse limits, *Theoret. Comput. Sci.* **87** (1991) 1–24.
[16] W. Taylor, Varieties of topological algebras, *J. Austral. Math. Soc. Ser. A* **23** (1977) 207–241.
[17] D. von Dantzig, Zur topologischen Algebra, *Math. Ann.* **107** (1933) 587–626.
[18] M. Wand, Final algebra semantics and data type extensions, *J. Comput. System Sci.* **19** (1981) 27–44.
[19] S. Warner, Topological Fields, Mathematics Studies, Vol. 157 (North Holland, Amsterdam, 1989).
[20] M. Wirsing, Algebraic specification, in: J. van Leeuwen ed., *Handbook of Theoretical Computer Science*, Vol. B, *Formal Models and Semantics* (Elsevier, Amsterdam, 1990) 675–788.
[21] K. Meinke and L.J. Steggles, Specification and verification in higher-order algebra: a case study of convolution, 189–222 in: J. Heering et al., eds., *HOA '93*, an International workshop on higher-order algebra, logic and term rewriting, Lecture Notes in Computer Science, Vol. 816 (Springer, Berlin, 1994).